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SHORT COMMUNICATION

A NOTE ON L_{-}^{p} CONVERGENCE OF CERTAIN COSINE SUMS

Maher MH Marzuq
84 Raymond Road. Plymouth, Massachusetts 02360, USA

ABSTRACT

In this paper we obtain theorems concerning L_{-}^{p} space with p=1, 0 < p < 1 and 0 < p < $\frac{1}{2}$. We will redefine some theorem of Telyakovskii (1973) and Corollary of Marzuq (1975) as well as Corollary of Ram (1977).

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AMS Subject Classifications 2010, Primary 42A20, 42A32

INTRODUCTION

Write

$$f(x):\ \frac{a_0}{2}+\sum_{k=1}^{\infty}a_k \cos kx\ , \tag{1.1}$$

$$f_m(x)=\frac{1}{2}\sum_{k=0}^{\infty}\Delta a_k+\sum_{k=1}^{m}\left(\sum_{j=k}^{m}\Delta a_j\right)\cos kx\ , \tag{1.2}$$

$$S_N(x)=\frac{a_0}{2}+\sum_{k=1}^{N}a_k \cos kx \tag{1.3}$$

$$g_n(x)=\frac{1}{2}\sum_{k=0}^{n}a_k+\sum_{k=1}^{n}\left(\sum_{j=k}^{n}a_j\right)\cos kx \tag{1.4}$$

Garrett and Stanojevic (1975), Garrett *et al*. (1980).

2. Statements of Results.

Definition 1. A sequence { a_n } is said to be of bounded variation if $\sum_{n=1}^{\infty}\left|\Delta a_n\right|<\infty$, where $\Delta a_n=a_n-a_{n+1}$.

Definition 2. A sequence { a_n } is said to be quasi-monotone if $a_n\to 0\ as\ n\to\infty$; $a_n>0$ ultimately and $\Delta a_n\geq -\delta_n$, where { δ_n } is a sequence of positive numbers Boas (1965).

Definition 3. A sequence { a_n }, n=1, 2…is said to satisfy condition S if

(i) $a_n\to 0\ as\ n\to\infty$,

(ii) there exist a numbers A_n such that $\{A_n\}$ is monotonically decreasing to 0 and $\sum_{n=1}^{\infty}A_n<\infty$ is convergent,

(iii) $\left|\Delta a_n\right|\leq A_n$ for all n.

Telyakovskii (1973).

Throughout this paper C denotes a positive constant, not necessarily the same at each occurrence.

We introduce the following definition.

Definition 4. A sequence { a_n }, n=1,2,……is said to satisfy condition T if

(i) $a_n\to 0\ as\ n\to\infty$,

(ii) there exist A_n n=1,2,….such that $\{A_n\}$, is a δ quasi-monotone sequence and $\sum_{n=1}^{\infty}n\delta_n$, $\sum_{n=1}^{\infty}A_n$ converge,

(iii) $\left|\Delta a_n\right|\leq A_n$ ultimately.

Marzuq (1982)

Independently Zenei (1992) considered the class $S(\delta)$, later it is proved by

Leindler (2000) and Telyakovskii (2000), the classes S,T and $S(\delta)$ are identical.

*Corresponding author email: maher_marzuq@yahoo.com

We prove:

Theorem 1. If $\{a_n\}$ belongs to the class T, then $f_n(x)$ converses to $f(x)$ in L^1_- norm.

Proof. Let $N^{/} > N$. Then by (1.3)

$$\left|S_{N^{/}}(x) - S_N(x)\right| = \left|\sum_{k=N+1}^{N^{/}} a_k \cos kx\right|,$$

and by partial summation,

$$\left|S_{N^{/}}(x) - S_N(x)\right| = \left|\sum_{k=N+1}^{N^{/}-1} (\Delta a_k) D_k(x) - a_{N+1} D_N(x) + a_{N^{/}} D_{N^{/}}(x)\right|$$

Where $D_k(x)$ is given by

$$D_k(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(n+\frac{1}{2})x}{2 \sin \frac{x}{2}} \tag{2.1}$$

Zygmund (1968).

We have

$$\left|D_k(x)\right| < \frac{\pi}{2x} \text{ for x > 0 }, \tag{2.2}$$

Bary (1964). Therefore for x.>0 and $N^{/}, N > N_0(\in)$,

$$\left|S_{N^{/}}(x) - S_N(x)\right| \le \frac{\pi}{2x}\left[\sum_{N+1}^{N^{/}-1} \left|\Delta a_k\right| + \left|a_{N+1}\right| + \left|a_{N^{/}}\right|\right].$$

Since $\{a_n\} \in T$, it follows that,

$\left|S_{N^{/}}(x) - S_N(x)\right| < \in$ for N, $N^{/} > N_0(\in)$ and x> 0.

Thus

$$f(x) = \lim_{N\to\infty} S_N(x) \tag{2.3}$$

exists for $x \in (0, \pi]$.

Theorem 2. Let $\{a_n\} \in T$. Then $f_m \to f$ in L^1_- norm.

Proof. Using partial summation on the right of (2.3) We get

$$f(x) = \lim_{N\to\infty}\left[\frac{a_0}{2} + \sum_{k=1}^{N-1} (\Delta a_k) D_k(x) + a_N D_N(x) - \frac{1}{2} a_1\right]$$

$$= \lim_{N\to\infty}\left[\sum_{k=0}^{N-1} (\Delta a_k) D_k(x) + a_N D_N(x)\right]$$

since $D_0(x) = \frac{1}{2}$. Then by (2.2) and Definition 4i, we get

$$f(x) = \sum_{k=0}^{\infty} (\Delta a_k) D_k(x) . \tag{2.4}$$

Now, by partial summation in the second term in (1.2) we obtain

$$f_m(x) = \sum_{k=0}^{m} (\Delta a_k) D_k(x) . \tag{2.5}$$

By applying partial summation again we have for m+1≤ n

$$\sum_{m+1}^{n} (\Delta a_k) D_k(x) = \sum_{m+1}^{n-1} (\Delta A_k) T_k(x) + A_n T_n(x) - A_{m+1} T_m(x) , \tag{2.6}$$

where $T_n(x) = \sum_{k=1}^{n} \frac{\Delta a_k}{A_k} D_k(x)$ Marzuq (1975)

Take $\alpha_k = \frac{\Delta a_k}{A_k}$.

Then for k sufficiently large $\left|\alpha_k\right| \le 1$, since $\{a_k\} \in T$.

Hence for m sufficiently large (2.6) gives

$$\int_0^{\pi}\left|\sum_{m+1}^{n} (\Delta a_k) D_k(x)\right| dx \le \sum_{m+1}^{n-1} \left|\Delta A_k\right| \int_0^{\pi} \left|T_k(x)\right| dx + A_n \int_0^{\pi} \left|T_n(x)\right| dx + A_{m+1} \int_0^{\pi} \left|T_m(x)\right| dx$$

By the definition of condition T and Telyakovskii (1973), we have

$$\int_0^{\pi}\left|\sum_{m+1}^{n} (\Delta a_k) D_k(x)\right| dx \le C\left[\sum_{m+1}^{n-1} \left|\Delta A_k\right| (k+1) + A_n(n+1) + A_{m+1}(m+1)\right]$$

Let $n \to \infty$, then by Boas (1965) with $\gamma = 1$

$$\underset{n\to\infty}{Lim} \int_0^{\pi}\left|\sum_{m+1}^{n} (\Delta a_k) D_k(x)\right| dx \le C\left[\sum_{m+1}^{\infty} \left|\Delta A_k\right| (k+1) + A_{m+1}(m+1)\right]$$

so that Fatou’s lemma implies

$$\int_0^{\pi}\left|\sum_{m+1}^{\infty} (\Delta a_k) D_k(x)\right| dx \le C\left[\sum_{m+1}^{\infty} \left|\Delta A_k\right| (k+1) + A_{m+1}(m+1)\right] \tag{2.7}$$

Now for sufficiently large m, (2.4), (2.5) and (2.7) give

$$\int_0^{\pi}|f(x)-f_m(x)|dx=\int_0^{\pi}\left|\sum_{m+1}^{\infty}(\Delta a_k)D_k(x)\right|dx\le C\left[\sum_{m+1}^{\infty}|\Delta A_k|(k+1)+A_{m+1}(m+1)\right]$$

and consequently,

$$\lim_{n\to\infty}\int_0^{\pi}|f(x)-f_m(x)|dx\le C\lim_{n\to\infty}\left[\sum_{k=m+1}^{\infty}|\Delta A_k|(k+1)+(m+1)A_{m+1}\right]=0$$

since $\sum_{k=1}^{\infty}(k+1)|\Delta A_k|<\infty$ and $\lim_{m\to\infty} mA_m=0$, by Boas (1965).

Thus $f_m\to f$ in L^1_norm.

3. Generalization of Telyakovskii Theorem. In view of the identity

$$f_n(x)=S_n(x)-a_{n+1}D_n(x), \qquad (3.1)$$

where $S_n(x)$ is give by (1.3) and f_n is given by (1.2) Marzuq (1975), we deduce the

following Corollary which is a part of Theorem 4 of

Telyakovskii (1973), Corollary of Marzuq (1975), and Corollary of Ram (1977).

Corollary 1. Let $\{a_n\}\in T$. Then (1.3) converses in L^1_norm to (1.1) if and only if

$a_n\log n\to 0$ as $n\to\infty$.

Proof. Let $a_n\log n\to 0$ as $n\to\infty$. Then by (3.1)

$$\int_0^{\pi}|f(x)-S_n(x)|dx=\le\int_0^{\pi}|f(x)-f_n(x)|dx+\int_0^{\pi}|f_n(x)-S_n(x)|dx$$

$$=\int_0^{\pi}|f(x)-f_n(x)|dx+|a_{n+1}|\int_0^{\pi}|D_n(x)|dx$$

$$\le\int_0^{\pi}|f(x)-f_n(x)|dx+C|a_{n+1}|\log n.$$

Since

$$\int_0^{\pi}|D_n(x)|dx : \frac{2}{\pi}\log n \qquad (3.2)$$

Zygmund (1968), then by the assumption, $a_n\log n$ and Theorem 1.

It follows that $S_n\to f$ L^1_norm.

Conversely, assume that $S_n\to f$ in L^1_norm, then (3.1) implies that

$$\int_0^{\pi}|a_{n+1}D_n(x)|dx=\int_0^{\pi}|f_n(x)-S_n(x)|dx\le\int_0^{\pi}|f(x)-S_n(x)|dx+\int_0^{\pi}|f(x)-f_n(x)|dx,$$

and hence the hypothesis on $a_n\log n$ and Theorem1 imply that

$$\int_0^{\pi}|a_{n+1}||D_n(x)|dx\to 0 \text{ as } n\to\infty.$$

Therefore (3.2) and the above result imply that $a_n\log n\to 0$ as $n\to\infty$.

This proves Corollary 1.

4. Conversance. In the space L^p $(0<p<1)$. We have the following theorem:

Theorem 2. Let $\{a_n\}$, n=0,1,…, be a sequence of bounded variation such that $a_n\to 0$ as $n\to\infty$. Then for 0< p < 1.

$$\lim_{n\to\infty}\int_0^{\pi}|f(x)-f_n(x)|dx=0.$$

Proof of Theorem 2. From (2.4) and (2.5) we have

$$|f(x)-f_n(x)|=\left|\sum_{n+1}^{\infty}(\Delta a_k)D_k(x)\right|,$$

so by (2.2) for x > 0

$$|f(x)-f_n(x)|\le\frac{\pi}{2x}\left(\sum_{n+1}^{\infty}|\Delta a_k|\right),$$

Consequently,

$$\lim_{n\to\infty}\int_0^{\pi}|f(x)-f_n(x)|^p dx\le C\lim_{n\to\infty}\left(\sum_{n+1}^{\infty}|\Delta a_k|\right)^p\int_0^{\pi}x^{-p}dx=0\cdot$$

Since

$\int_0^{\pi}x^{-p}dx<\infty$ for o < p < 1 and $\{a_k\}$, k = 0,1,…, is a sequence of bounded variation.

This proves Theorem 2.

By considering $g_n(x)$ given by (1.4), and $g(x)=\lim_{n\to\infty}g_n(x)$ exist Marzuq (2005), where $\{a_n\}$ is a sequence of bounded variation and $\lim_{n\to\infty}a_n=0$ as $n\to\infty$.

We find that p has to be restricted to (0, $\frac{1}{2}$). In this case we have the following result:

Theorem 3. Let $\{a_n\}$, n = 0,1,…, be a sequence of bounded variation such that $a_n \to 0$ as $n \to \infty$. Then for 0 < p < $\frac{1}{2}$,

$$\lim_{n\to\infty}\int_0^{\pi}\left|g(x)-g_n(x)\right|^p dx = 0 .$$

Proof. By partial summation (1.4) gives

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} a_k + \sum_{k=1}^{n-1} a_k D_k(x) - \frac{1}{2}\sum_{k=1}^{n} a_k + a_n D_n(x)$$

$$= \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k D_k(x) , \qquad (4.1)$$

Where $D_k(x)$ is give by (2.1).

Apply partial summation to the right side of (4.1). We get

$$g_n(x) = \frac{1}{2}a_o + \sum_{k=1}^{n-1}(\Delta a_k)(k+1)F_k(x) + a_n(n+1)F_n(x) - \frac{1}{2}a_1$$
$$= \sum_{k=0}^{n-1}(\Delta a_k)(k+1)F_k(x) + a_n(n+1)F_n(x) , \qquad (4.2)$$

where

$$F_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) .$$

Since

$$F_n(x) \le \frac{C}{(n+1)x^2} \text{ , } 0 < x \le \pi \qquad (4.3)$$

Zygmund (1968) and $a_n \to 0$ as $n \to \infty$, (4.2) gives

$$g(x) = \lim_{n\to\infty} g_n(x) = \sum_{k=0}^{\infty}(\Delta a_k)(k+1)F_k(x) . \quad (4.4)$$

Thus (4.2), (4.3) and (4.4) give

$$\left|g(x) - g_n(x)\right| \le \frac{C}{x^2}\left(\sum_{k=n}^{\infty}\left|\Delta a_k\right| + \left|a_n\right|\right) .$$

Raise both sides to the pth power and integrate over (0, π) , and take the limit as is $n \to \infty$. We obtain

$$\lim_{n\to\infty}\int_0^{\pi}\left|g(x)-g_n(x)\right|^p dx \le C\lim_{n\to\infty}\left(\sum_{k=n}^{\infty}\left|\Delta a_k\right| + \left|a_n\right|\right)^p \int_0^{\pi} x^{-2p}dx = 0,$$

since $\int_0^{\pi} x^{-2p}dx$ is finite for 0 < p < $\frac{1}{2}$. This proves theorem 3.

We need the following inequality

$$(a+b)^p \le 2^p(a^b + b^p) , \quad a \ge 0 , \quad b \ge 0 , \qquad (4.5)$$

for 0 < p < ∞ Duren (1970).

Corollary 2. If $\{a_n\}$, n = 0, 1, …, is a sequence of bounded variation such that $a_n \to 0$ as $n \to \infty$, then $g \in L^p[0, \pi]$ for 0 < p < $\frac{1}{2}$.

Proof. Write

$g(x) = g(x) - g_n(x) + g_n(x)$, then by inequality (4.5),

$$\left|g(x)\right|^p \le 2^p\left(\left|g(x)-g_n(x)\right|^p + \left|g_n(x)\right|^p\right) . \quad (4.6)$$

By (4.2) and (4.3) we have

$$\left|g_n(x)\right| \le \frac{C}{x^2}\sum_{k=0}^{n-1}\left|\Delta a_k\right| + \frac{C}{x^2}\left|a_n\right| .$$

Hence by using (4.5) again, (4.6) becomes

$$\left|g(x)\right|^p \le 2^p\left\{\left|g(x)-g_n(x)\right|^p + 2^p\left[\frac{C}{x^{2p}}\left(\sum_{k=0}^{n-1}\left|\Delta a_k\right|\right)^p + \frac{C}{x^{2p}}\left|a_n\right|^p\right]\right\}.$$

Thus

$$\int_0^{\pi}\left|g(x)\right|^p dx \le 2^p\int_0^{\pi}\left|g(x)-g_n(x)\right|^p dx + 2^p C\left[\left(\sum_{k=0}^{n-1}\left|\Delta a_k\right|\right)^p + \left|a_n\right|^p\right]\int_0^{\pi} x^{-2p}dx$$

Let $n \to \infty$. Then by Theorem 3 and the hypothesis of the Corollary we conclude that

$$\int_0^{\pi}\left|g(x)\right|^p dx \le C\left(\sum_{k=0}^{\infty}\left|\Delta a_k\right|\right)^p < \infty .$$

Thus $g \in L^p[0,\pi]$, 0 < p < $\frac{1}{2}$.

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REFERENCES

Bary, NK. 1964. A Treatise on Trigonometric series, Vol. II, Pergamon Press, Oxford.

Boas, RP Jr. 1965. Quasi-positive sequences and trigonometric series, Proc. Lond. Math. Soc. (3)14A :38-46.

Duren, PL. 1970. Theory of H^p Spaces, Academic Press, NY, USA.

Garrett, JW. and Stanojevic. 1975. On Integrability and L^1 convergence of certain cosine sums, Notes Amer. Math. Soc. 22(A-166).

Garrett, JW., Rees, CS. and Stanojevic, CV. 1980. L^1-Convergence of Fourier series with coefficients of bounded variation, Proc. Amer. Math. Soc. 80:423-430.

Leindler, L. 2000. On the equivalence of class of numerical sequences, Analysis Mathematica (Szeged-Moscow). 26:227-234.

Marzuq, M. 1975. L^1-Convergence of certain cosine sums, J. Univ. Kuwait (Sci) 2 :9-10.

Marzuq, M. 2005. Integrability theorems of trigonometric series, Toyama Math. J. 28:1-9.

Maher, MH. Marzuq. 1982. Properties of Functions on Bounded Star-Shaped Circular Domains in $C^N(N>1)$. Thesis, Department of Mathematics, State University of New York at Buffalo, New York, USA. 1-193.

Ram, B. 1977. Convergence of certain sums in the metric space L, Proc. Amer. Math. Soc. 66:258-260.

Telyakovskii, SA. 1973. Concerning a sufficient condition of Sidon for the Integrability of trigonometric series. Math. Notes. 14:742-748.

Telyakovskii, SA. 2000. A remark on condition for Integrability of trigonometric series. Moscow University Bulletin. 4:58-60(in Russian).

Zenei, SZA. 1990. Integrability of trigonometric series, Tamkang J. Math. 21:295-301.

Zygmund, A. 1968. Trigonometric Series, Cambridge Univ. Press, (vol. I & II), New York, USA.

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