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AN APPROXIMATE SOLUTION OF HYPERSINGULAR AND SINGULAR INTEGRAL EQUATIONS

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ABSTRACT

This paper is devoted to study the approximate solution of hypersingular and singular integral equations by means of chebyshev polynomial of second kind. Some examples are presented to illustrate the method.

Keywords: Hypersingular and singular integral equations, cauchy kernel, chebyshev polynomial.
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INTRODUCTION

Many important problems of engineering mechanics like elasticity, plasticity, and fracture mechanics and aerodynamics can be reduced to the solution of finite-part or hypersingular integral equations see Chan *et al*. (2003), Ladopoulous (2000) and Manegato (2009). Hence, it is interest to solve numerically this type of singular integral equations Boykov *et al*. (2010), Mandal and Bera (2006). Chebyshev polynomials are of great importance in many areas of mathematics particularly approximation theory, see Akyuz- Dascioglu and Cerdik (2006) and Mandal and Bera, (2006).

In this paper we analyzed the numerical solution of hypersingular and singular integral equations by using Chebyshev polynomial of second kind to obtain systems of linear algebraic equations, these systems are solved numerically. The methodology of the present work expected to be useful for solving hypersingular and singular integral equations of the first kind, involving partly hypersingular and singular kernels respectively and partly regular kernels are developed here. The singularity in singular integral equation is assumed to be of the Cauchy type. The method is illustrated by considering some examples.

Consider the following hypersingular integral equation of first kind, over a finite interval:

$$\int_{-1}^{1} x(\tau)\left[\frac{h(t,\tau)}{(\tau-t)^2}+M(t,\tau)\right]d\tau = f(t), \qquad -1\le t\le 1 \quad (1.1)$$

with $x(\pm 1)=0$, where $h(t,\tau)$, $M(t,\tau)$ and $f(t)$ are given real-valued continuous functions belong to the class Holder of continuous functions, $x(t)$ is unknown function to be determine. The hypersingular integral equations of form (1.1) and other different forms have many applications in Banerjea *et al*. (1996), Chan *et al*. (2003), Kanoria and Mandal) 2002 and Parsons and Martin (1992, 1994). An approximate method for solving (1.1) using a polynomial approximation of degree n has been proposed in Mandal and Bera (2006).

Singular integral equation of first kind, with a Cauchy type singular kernel, over a finite interval can be represented by

$$\int_{-1}^{1} x(\tau)\left[\frac{\tilde{h}(t,\tau)}{\tau-t}+\tilde{M}(t,\tau)\right]d\tau = g(t), \qquad -1< t< 1 \quad (1.2)$$

where $\tilde{h}(t,\tau)$, $\tilde{M}(t,\tau)$ and $g(t)$ are given real-valued continuous functions belong to the class Holder of continuous functions and $\tilde{h}(t,t)\neq 0$. In equation (1.2) the singular kernel is interpreted as Cauchy principle value. Integral equation of form (1.2) and other different forms occur in varieties of mixed boundary value problems of mathematical physics which include problems of two dimensional deformations of isotropic elastic bodies involving cracks (Gakhov, 1966; Ladopoulous, 2000; Martin and Rizzo, 1989) and scattering of two-dimensional surface water waves by vertical barriers (Chakrabarti, 1989; Chakrabarti and Bharatti, 1992) and other related problems. An approximate method for solving (1.2) using a polynomial approximation of degree n has been proposed in Chakrabarti and Berghe (2004).

The analytical solution of the simple singular integral equation

$$\int_{-1}^{1} \frac{x(\tau)}{\tau-t}d\tau = g(t), \qquad -1<t<1 \qquad (1.3)$$

For $\tilde{h}(t,\tau)=1$ and $\tilde{M}(t,\tau)=0$, bounded at the end points $t=\pm 1$, is given by the following formula Eshkuvatov *et al*. (2009):

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$$x(t)=-\frac{\sqrt{1-t^2}}{\pi^2}\int_{-1}^{1}\frac{g(\tau)}{\sqrt{1-\tau^2}\,(\tau-t)}d\tau \qquad (1.4)$$

under the condition

$$\int_{-1}^{1}\frac{g(\tau)}{\sqrt{1-\tau^2}}d\tau=0. \qquad (1.5)$$

Let us show that singular integral equations of the first kind under some set of additional conditions can be reduced to hypersingular integral equations. For example, let us consider the equation

$$\int_{-1}^{1}\frac{\tilde{h}(t,\tau)x(\tau)}{\tau-t}d\tau=g(t), \qquad (1.6)$$

where $\tilde{h}(t,\tau)$ and *g(t)* are continuously differentiable functions. If equation (1.6) has a solution *x(t)* then the following equation

$$\int_{-1}^{1}\frac{\tilde{h}_t'(t,\tau)x(\tau)}{\tau-t}d\tau+\int_{-1}^{1}\frac{h(t,\tau)x(\tau)}{(\tau-t)^2}d\tau=g'(t) \qquad (1.7)$$

has the same solution. However, equation. (1.7) may also have additional solutions.

A check that the obtained solution of equation (1.7) is also a solution of equation (1.6) presents no difficulties and therefore the transition to high-order singularity equations is very attractive in numerical implementation (Boykov *et al*., 2010). In fact if $\varphi_1(\tau)=x(\tau)\,h(t,.)$, the hypersingular integral in (1.1) is defined as:

$$\frac{d}{dt}\int_{-1}^{1}\frac{\varphi_1(\tau)}{\tau-t}d\tau=\frac{d}{dt}\left[\varphi_1(1)\ln|1-t|-\varphi_1(-1)\ln|1+t|-\int_{-1}^{1}\varphi_1(\tau)\ln|\tau-t|d\tau\right]$$

$$=-\frac{\varphi_1(1)}{1-t}-\frac{\varphi_1(-1)}{1+t}+\left(\left.\frac{\varphi_1(\tau)}{\tau-t}\right|_{-1}^{1}+\int_{-1}^{1}\frac{\varphi_1(\tau)}{(\tau-t)^2}d\tau\right)=\int_{-1}^{1}\frac{\varphi_1(\tau)}{(\tau-t)^2}d\tau; \qquad -1<t<1$$

Thus

$$\frac{d}{dt}\int_{-1}^{1}\frac{\varphi_1(\tau)}{\tau-t}d\tau=\int_{-1}^{1}\frac{\varphi_1(\tau)}{(\tau-t)^2}d\tau; \qquad (1.8)$$

In this paper the used approximate method for solving equations (1.1) and (1.2) stems from recent work Eshkuvatov *et al*. (2009) wherein an approximate method has been developed to solve the simple equation (1.3). The approximate method developed below appears to be quite appropriate for solving the most general type equations (1.1) and (1.2). Also we illustrate by some examples that the approximate solutions of the singular integral equations and the corresponding hypersingular singular integral equations are coincided.

The approximate solution

The chebyshev polynomials of the second kind, U_i, can be defined by the recurrence relation

$$\left.\begin{array}{ll} U_0(t)=1, & U_1(t)=2t \\ U_n(t)=2tU_{n-1}(t)-U_{n-2}(t) & n\ge 2\end{array}\right\} \qquad (2.1)$$

Let the unknown function $x(t)$ in Eq. (1.1) be approximated by the polynomial function x_n :

$$x_n(t)=\sqrt{1-t^2}\sum_{i=0}^{n}a_iU_i(t), \qquad -1\le t\le 1 \qquad (2.2)$$

where a_i, $i=0,1,2,...,n$ are unknown coefficients.

Substituting the approximate solution (2.2) for the unknown function into (1.1), where $h(t,\tau)\cong\sum_{p=0}^{m}h_p(t)\tau^p$, $M(t,x)\cong\sum_{q=0}^{s}M_q(t)\tau^q$, we obtain

$$\sum_{i=0}^{n}a_i\lambda_i(t)=f(t), \qquad -1\le t\le 1, \qquad (2.3)$$

where

$$\lambda_i(t)=\sum_{p=0}^{m}h_p(t)v_{p,i}(t)+\sum_{q=0}^{s}M_q(t)\zeta_{q,i} \qquad (2.4)$$

with

$$v_{i,p}(t)=\int_{-1}^{1}\frac{\tau^p\sqrt{1-\tau^2}\,U_i(\tau)}{(\tau-t)^2}d\tau \qquad -1\le t\le 1, \qquad (2.5)$$

$$\zeta_{q,i}=\int_{-1}^{1}\tau^q\sqrt{1-\tau^2}\,U_i(\tau)\,d\tau \qquad (2.6)$$

Using the zeros t_k of the chebyshev polynomial of the first kind $T_{n+1}(t)$ we obtain the following system of linear equations :

$$\sum_{i=0}^{n}a_i\left(\sum_{p=0}^{m}h_p(t_k)v_{p,i}(t_k)+\sum_{q=0}^{s}M_q(t_k)\zeta_{q,i}\right)=f(t_k), \quad (k=1,2,...,n+1), \qquad (2.7)$$

Where

$$t_k=\cos\left(\frac{(2k-1)\pi}{2(n+2)}\right), \quad k=1,2,...,n+1, \qquad (2.8)$$

By solving the system of equations (2.7) for the unknown coefficients a_i, $i=0,1,...,n$ and substituting the values of a_i into (2.2) we obtain the approximate solution of equation (1.1).

Similarly, substituting the approximate solution (2.2) for the unknown function into (1.2), such that $\tilde{h}(t,\tau) \cong \sum_{p=0}^{m} \tilde{h}_p(t)\tau^p$, $\tilde{M}(t,x) \cong \sum_{q=0}^{s} \tilde{M}_q(t)\tau^q$,

yields

$$\sum_{i=0}^{n} \tilde{a}_i \beta_i(t) = g(t), \qquad -1 < t < 1, \qquad (2.9)$$

where

$$\beta_i(t) = \sum_{p=0}^{m} \tilde{h}_p(t)\, w_{i,p}(t) + \sum_{q=0}^{s} \tilde{M}_q(t)\,\zeta_{q,i} \qquad (2.10)$$

with

$$w_{i,p}(t) = \int_{-1}^{1} \frac{\tau^p \sqrt{1-\tau^2}\, U_i(\tau)}{\tau - t} d\tau \quad -1 < t < 1, \qquad (2.11)$$

Substituting the collocation points t_k into (2.9) we obtain the following system of linear equations :

$$\sum_{i=0}^{n} \tilde{a}_i \left(\sum_{p=0}^{m} \tilde{h}_p(t_k) w_{p,i}(t_k) + \sum_{q=0}^{s} \tilde{M}_q(t_k)\zeta_{q,i} \right) = g(t_k), \quad (k = 1,2,\ldots,n+1), \qquad (2.12)$$

By solving the system of equations (2.12) for the unknown coefficients $\tilde{a}_i$, $i = 0,1,\ldots,n$ and substituting the values of $\tilde{a}_i$ into (2.2), instead of the values a_i, we obtain the approximate solution of equation (1.2) as the form

$$x_n(t) \cong \sqrt{1-t^2} \sum_{i=0}^{n} \tilde{a}_i U_i(t) \quad -1 < t < 1 \qquad (2.13)$$

Numerical examples

In this section, we consider six problems to illustrate the above method. All results were computed using FORTRAN code.

Example 1 Consider the following singular integral equation

$$\int_{-1}^{1} \frac{(\tau + 2t)x(\tau)}{\tau - t} d\tau = 6t^3 - 3t, \ -1 < t < 1. \qquad (3.1)$$

And seek the solution $x(\tau)$ as the polynomial function x_n :

$$x_n(t) = \sqrt{1-t^2} \sum_{i=0}^{3} \tilde{a}_i U_i(t), \quad -1 < x < 1 \qquad (3.2)$$

Where

$$\tilde{h}(t,\tau) = \tau + 2t, \quad \tilde{M}(t,\tau) = 0, \quad g(t) = 6t^3 - 3t.$$

So, one gets

$$\tilde{h}_0(t) = 2t, \ \tilde{h}_1(t) = 1, \ \tilde{h}_p(t) = 0; \ (p \geq 2), \ \tilde{M}_q(t) = 0; \ (q \geq 0)$$

Hence the relation (2.9) takes the form:

$$\beta_i(t) = 2t w_{0,i}(t) + w_{1,i}(t) \qquad (3.3)$$

where

$$w_{0,i}(t) = \int_{-1}^{1} \frac{\sqrt{1-\tau^2}\, U_i(\tau)}{\tau - t} d\tau \quad -1 < t < 1, \qquad (3.4)$$

$$w_{1,i}(t) = \int_{-1}^{1} \frac{\tau\sqrt{1-\tau^2}\, U_i(\tau)}{\tau - t} d\tau \quad -1 < t < 1, \qquad (3.5)$$

and applying the relation

$$\int_{-1}^{1} \frac{\sqrt{1-\tau^2}\, U_i(\tau)}{\tau - t} d\tau = -\pi T_{i+1}(t) \qquad (3.6)$$

Where $T_{i+1}(t)$ is the Chebyshev polynomial of first kind, The relations (3.4) –(3.6) we get

$$\beta_i(t) = \begin{cases} -\pi\left(3t^2 - \frac{1}{2}\right) & i = 0 \\ -3\pi(2t^3 - t) & i = 1 \\ -3\pi(4t^4 - 3t^2) & i = 2 \\ -\pi\left(24t^5 - 16t^3 + \frac{7}{2}t\right) & i = 3 \end{cases} \qquad (3.7)$$

By using the zeros t_k of Chebyshev polynomial $T_{n+1}(t)$, for $n = 3$, we obtain the following system of linear equations :

$$\sum_{i=0}^{3} \tilde{a}_i \beta_i(t_k) = g(t_k), \quad (k = 1,2,3,4),$$

Where $\beta(t_k) = \beta(t)$ at $t = t_k$. By solving this system for the unknown coefficients $\tilde{a}_i$, $i = 0,1,2,3$ that produces

$$\left.\begin{array}{ll} \tilde{a}_0 = 2.455668 \times 10^{-8}, & \tilde{a}_1 = -3.183099 \times 10^{-1}, \\ \tilde{a}_2 = -7.818176 \times 10^{-9}, & \tilde{a}_3 = -1.457889 \times 10^{-9} \end{array}\right\} \qquad (3.8)$$

From (2.1), (3.8) we obtain the approximate solution of equation (3.1) in the form

$$x_n(t) \cong \frac{-2}{\pi}\left(t\sqrt{1-t^2}\right) \tag{3.9}$$

Which coincides with the exact solution. The error of approximate solution (3.9) of equation (3.1) at $n = 20$ is given by table 1.

In example 2 we solve the corresponding hypersingular integral equation of the equation (3.1).

Example 2 Consider the following hypersingular integral equation

$$\int_{-1}^{1} \frac{x(\tau)}{(\tau-t)^2}\, d\tau = 4t, \quad -1 \le t \le 1 \tag{3.10}$$

And seek the solution $x(\tau)$ as the polynomial function x_n :

$$x_n(t) = \sqrt{1-t^2}\sum_{i=0}^{3} a_i U_i(t), \quad -1 \le x \le 1 \tag{3.11}$$

Where $h(t,\tau)=1, \quad M(t,\tau)=0, \quad f(t)=4t$. So, one gets

$h_0(t)=1,\ h_p(t)=0;\ (p \ge 1), M_q(t)=0;\ (q \ge 0)$.

Hence the relation (2.4) takes the form:

$$\lambda_i(t) = v_{0,i}(t) \tag{3.12}$$

where

$$v_{0,i}(t) = \int_{-1}^{1} \frac{\sqrt{1-\tau^2}\, U_i(\tau)}{(\tau-t)^2}\, d\tau \quad -1 \le t \le 1, \tag{3.13}$$

Applying the relation

$$\int_{-1}^{1} \frac{\sqrt{1-\tau^2}\, U_i(\tau)}{(\tau-t)^2}\, d\tau = -\pi(i+1)U_i(t) \tag{3.14}$$

From the relations (3.12)- (3.14) we obtain

$$\lambda_i(t) = \begin{cases} -\pi & i=0 \\ -4\pi t & i=1 \\ -\pi\left(4t^3-2t-1\right) & i=2 \\ -\pi\left(8t^4-4t^2-8t-1\right) & i=3 \end{cases} \tag{315}$$

By using the zeros t_k of Chebyshev polynomial $T_{n+1}(t)$, for $n = 3$, we obtain the following system of linear equations :

$$\sum_{i=0}^{3} a_i \lambda_i(t_k) = f(t_k), \quad (k = 1,2,3,4),$$

Where $\lambda(t_k) = \lambda(t)$ at $t = t_k$. By solving this system for the unknown coefficients $a_i,\ i = 0,1,2,3$ that produces

$$\left.\begin{array}{ll} a_0 = 1.038365 \times 10^{-7}, & a_1 = -3.183098 \times 10^{-1}, \\ a_2 = 9.839027 \times 10^{-10}, & a_3 = -1.743203 \times 10^{-9} \end{array}\right\} \tag{3.16}$$

From (2.1) and (3.16) we obtain the approximate solution of equation (3.10) in the form

$$x_n(t) \cong \frac{-2}{\pi}\left(t\sqrt{1-t^2}\right) \tag{3.17}$$

Which coincides with the exact solution. The error of approximate solution (3.17) of equation (3.10) at $n = 20$ is given by Table 1.

Example 3 Consider the following singular integral equation

$$\int_{-1}^{1} x(\tau)\left[\frac{\left(\tau^2+t\right)}{\tau-t} + \left(\tau^3+t^2\right)\right] d\tau = 2t^4 - 2t^2 - \frac{3}{8}, \ -1 < t < 1 \tag{3.18}$$

Where

$\tilde{h}(t,\tau) = \tau^2 + t, \quad \tilde{M}(t,\tau) = \tau^3 + t^2, \quad g(t) = 2t^4 - 2t^2 - \frac{3}{8}\cdot$

So, one gets

$\tilde{h}_0(t) = t,\ \tilde{h}_1(t) = 0,\ \tilde{h}_2(t) = 1,\ \tilde{h}_p(t) = 0;\ (p \ge 3),$

$\tilde{M}_0(t) = t^2,\ \tilde{M}_1(t) = 0,\ \tilde{M}_2(t) = 0,\ \tilde{M}_3(t) = 1,\ \tilde{M}_q(t) = 0;\ (q \ge 4)$

.

Hence the relation (2.9) takes the form:

$$\beta_i(t) = t w_{0,i}(t) + w_{2,i}(t) + t^2\zeta_{0,i} + \zeta_{3,i} \tag{3.19}$$

where

$$w_{2,i}(t) = \int_{-1}^{1} \frac{\tau^2\sqrt{1-\tau^2}\, U_i(\tau)}{\tau-t}\, d\tau \quad -1 < t < 1, \tag{3.20}$$

$$\zeta_{0,i} = \int_{-1}^{1} \sqrt{1-\tau^2}\, U_i(\tau)\, d\tau \tag{3.21}$$

$$\zeta_{3,i} = \int_{-1}^{1} \tau^3\sqrt{1-\tau^2}\, U_i(\tau)\, d\tau \tag{3.22}$$

By applying the orthogonal property

$$\int_{-1}^{1} \sqrt{1-\tau^2}\, U_i(\tau) U_i(\tau)\, d\tau = \begin{cases} 0 & i=0 \\ \dfrac{\pi}{2} & i=j \end{cases} \tag{3.23}$$

From the relations (3.4) and (3.19)-(3.23) we get

$$\beta_i(t)=\begin{cases}-\dfrac{\pi}{2}\left(2t^3+t^2-t\right) & i=0\\ -\pi\left(2t^4+2t^3-t^2-t-\dfrac{3}{8}\right) & i=1\\ -\pi\left(4t^5+4t^4-3t^3-3t^2+\dfrac{1}{4}t\right) & i=2\\ -\pi\left(8t^6+8t^5-8t^4-8t^3+t^2+t-\dfrac{1}{16}\right) & i=3\end{cases} \quad (3.24)$$

By using the zeros t_k of Chebyshev polynomial $T_{n+1}(t)$, for $n=3$, we obtain the following system of linear equations :

$$\sum_{i=0}^{3}\tilde{a}_i\beta_i(t_k)=g(t_k),\quad (k=1,2,3,4),$$

By solving this system for the unknown coefficients $\tilde{a}_i$, $i=0,1,2,3$ that produces

$$\left.\begin{aligned}&\tilde{a}_0=6.366197\times10^{-1}, \quad \tilde{a}_1=-3.183099\times10^{-1},\\ &\tilde{a}_2=2.279989\times10^{-8}, \quad \tilde{a}_3=-7.819254\times10^{-9}\end{aligned}\right\} \quad (3.25)$$

From (2.1), (3.25) we obtain the approximate solution of equation (3.18) in the form

$$x_n(t)\cong\frac{2\sqrt{1-t^2}}{\pi}(1-t) \quad (3.26)$$

Which coincides with the exact solution. The error of approximate solution (3.26) of equation (3.18) at $n=20$ is given by table 1.

In example 4 we solve the corresponding hypersingular integral equation of the equation (3.18).

Example 4 Consider the following hypersingular integral equation

$$\int_{-1}^{1}x(\tau)\left[\frac{(\tau^2+t)}{(\tau-t)^2}+2t\right]d\tau=8t^3-2t^2-2t+1,\quad -1\le t\le 1 \quad (3.27)$$

Where

$h(t,\tau)=\tau^2+t, M(t,\tau)=2t,\ f(t)=8t^3-2t^2-2t+1$

So, one gets

$$h_0(t)=t,\quad h_1(t)=0,\quad h_2(t)=1,\quad h_p(t)=0;\ (p\ge3),$$
$$M_0(t)=2t,\quad M_q(t)=0;\ (q\ge1)$$

Hence the relation (2.4) takes the form:

$$\lambda_i(t)=tv_{0,i}(t)+v_{2,i}(t)+2t\zeta_{0,i} \quad (3.28)$$

where

$$v_{2,i}(t)=\int_{-1}^{1}\frac{\tau^2\sqrt{1-\tau^2}\,U_i(\tau)}{(\tau-t)^2}d\tau,\ -1\le t\le1, \quad (3.29)$$

From the relations (3.13), (3.21), (3.28) and (3.29) we get

$$\lambda_i(t)=\begin{cases}\pi\left(\dfrac{1}{2}-3t^2\right) & i=0\\ -\pi\left(8t^3+4t^2-2t\right) & i=1\\ -\pi\left(20t^4+12t^3-9t^2-3t\right) & i=2\\ -\pi\left(48t^5+32t^4-32t^3-16t^2+\dfrac{9}{2}t\right) & i=3\end{cases} \quad (3.30)$$

By using the zeros t_k of Chebyshev polynomial $T_{n+1}(t)$, for $n=3$, we obtain the following system of linear equations :

$$\sum_{i=0}^{3}a_i\lambda_i(t_k)=f(t_k),\quad (k=1,2,3,4),$$

By solving this system for the unknown coefficients a_i, $i=0,1,2,3$ that produces

$$\left.\begin{aligned}&a_0=6.366197\times10^{-1}, \quad a_1=-3.183098\times10^{-1},\\ &a_2=-7.354522\times10^{-8}, \quad a_3=2.346384\times10^{-8}\end{aligned}\right\} \quad (3.31)$$

From (2.1), (3.31) we obtain the approximate solution of equation (3.27) in the form

$$x_n(t)\cong\frac{2\sqrt{1-t^2}}{\pi}(1-t) \quad (3.32)$$

Which coincides with the exact solution. The error of approximate solution (3.32) of equation (3.27) at $n=20$ is given by table 2.

Example 5. Consider the following singular integral equation

$$\int_{-1}^{1}\frac{x(\tau)}{\tau-t}d\tau+\int_{-1}^{1}(\tau^2+t^2)\,x(t)\,dt=\frac{-3}{2}t^2+2t \tag{3.33}$$

Which corresponds with $\tilde{h}(\tau,t)=1$ and $\tilde{M}(\tau,t)=\tau^2+t^2$. So, one gets

$$\tilde{h}_0(t)=1,\quad \tilde{h}_p(t)=0,\qquad (p>0)$$

$$\tilde{M}_0(t)=t^2,\ \tilde{M}_1(t)=0,\ \tilde{M}_2(x)=1,\ \tilde{M}_q(t)=0\ \ (q>2)$$

Hence, we get

$$\sum_{i=0}^{3}\tilde{a}_i\beta_i(t)=\frac{-3}{2}t^2+2t,\qquad -1<x<1 \tag{3.34}$$

where

$$\beta_i(t)=w_{0,i}(t)+t^2\zeta_{0,i}+\zeta_{2,i},\quad (i=0,1,2,3,4) \tag{3.35}$$

with

$$\zeta_{2,i}=\int_{-1}^{1}\tau^2\sqrt{1-\tau^2}U_i(\tau)\,d\tau \tag{3.36}$$

from (3.4),(3.21), (3.35) and (3.36) we get

$$\beta_i(t)=\begin{cases}\frac{\pi}{8}\left(4t^2-8t+1\right) & i=0\\ -\pi\left(2t^2-1\right) & i=1\\ \frac{-\pi}{8}\left(32t^3-24t-1\right) & i=2\\ -\pi\left(8t^4-8t^2+1\right) & i=3\end{cases} \tag{3.37}$$

By using the zeros t_k of Chebyshev polynomial $T_{n+1}(t)$, for $n=3$, we obtain the following system of linear equations :

$$\sum_{i=0}^{3}\tilde{a}_i\beta_i(t_k)=g(t_k),\quad (k=1,2,3,4),$$

By solving this system for the unknown coefficients $\tilde{a}_i$, $i=0,1,2,3$ that produces

$$\left.\begin{array}{ll}\tilde{a}_0=-6.366197\times10^{-1}, & \tilde{a}_1=7.957754\times10^{-2},\\ \tilde{a}_2=1.746461\times10^{-8}, & \tilde{a}_3=1.827517\times10^{-8}\end{array}\right\} \tag{3.38}$$

From (2.1), (3.38) we obtain the approximate solution of equation (3.33) in the form

$$x_n(t)\cong\frac{-\sqrt{1-t^2}}{2\pi}\left(4-t\right) \tag{3.39}$$

Which coincides with the exact solution. The error of approximate solution (3.39) of equation (3.33) at $n=20$ is given by table 2.

In example 6 we solve the corresponding hypersingular integral equation of the equation (3.33).

Example 6 Consider the following hypersingular integral equation

$$\int_{-1}^{1}\frac{x(\tau)}{(\tau-t)^2}d\tau+\int_{-1}^{1}2t\,x(\tau)\,d\tau=-3t+2,\qquad -1\le t\le1 \tag{3.40}$$

Which corresponds with $h(t,\tau)=1$ and $L(t,\tau)=2t$. So, one gets

$$h_0(t)=1,\quad h_p(t)=0,\ \ (p>0)$$

and

$$M_0(t)=2t,\quad M_q(t)=0\ \ (q>0)$$

Hence the relation (2.4) takes the form:

$$\lambda_i(t)=v_{0,i}(t)+2t\zeta_{0,i} \tag{3.41}$$

from (3.13) and (3.21) we get

$$\lambda_i(t)=\begin{cases}\pi(t-1) & i=0\\ -4\pi t & i=1\\ -3\pi\left(4t^2-1\right) & i=2\\ -16\pi\left(2t^3-t\right) & i=3\end{cases} \tag{3.42}$$

By using the zeros of Chebyshev polynomial $T_{n+1}(t)$, for $n=3$, we obtain the following system of linear equations :

$$\sum_{i=0}^{3}a_i\lambda_i(t_k)=f(t_k),\quad (k=1,2,3,4),$$

By solving this system for the unknown coefficients a_i, $i=0,1,2,3$ that produces

$$\left.\begin{array}{ll}a_0=-6.366199\times10^{-1}, & a_1=7.957745\times10^{-2},\\ a_2=-2.038681\times10^{-8}, & a_3=2.170145\times10^{-8}\end{array}\right\} \tag{3.43}$$

From (2.1), (3.43) we obtain the approximate solution of equation (3.40) in the form

Table 1. Illustrates the errors of the approximate solutions (3.9), (3.17) and (3.26) respectively at $n = 20$.

x	Error 1	Error 2	Error 3
-.950000E+00	.298023E-07	.298023E-07	.000000E+00
-.900000E+00	.149012E-07	.596046E-07	.000000E+00
-.700000E+00	.298023E-07	.596046E-07	.000000E+00
-.500000E+00	.298023E-07	.596046E-07	.596046E-07
-.300000E+00	.447035E-07	.447035E-07	.596046E-07
-.100000E+00	.372529E-07	.149012E-07	.596046E-07
.000000E+00	.323749E-07	.000000E+00	.596046E-07
.100000E+00	.298023E-07	.149012E-07	.596046E-07
.300000E+00	.149012E-07	.447035E-07	.894070E-07
.500000E+00	.000000E+00	.596046E-07	.894070E-07
.700000E+00	.000000E+00	.596046E-07	.745058E-07
.900000E+00	.149012E-07	.596046E-07	.502914E-07
.950000E+00	.149012E-07	.298023E-07	.363216E-07

Table 2. Illustrates the errors of the approximate solutions (3.32), (3.39) and (3.44) respectively at $n = 20$.

x	Error 4	Error 5	Error 6
-.950000E+00	.596046E-07	.298023E-07	.447035E-07
-.900000E+00	.596046E-07	.298023E-07	.596046E-07
-.700000E+00	.119209E-06	.000000E+00	.119209E-06
-.500000E+00	.119209E-06	.000000E+00	.119209E-06
-.300000E+00	.119209E-06	.000000E+00	.178814E-06
-.100000E+00	.596046E-07	.596046E-07	.119209E-06
.000000E+00	.596046E-07	.596046E-07	.178814E-06
.100000E+00	.000000E+00	.596046E-07	.178814E-06
.300000E+00	.298023E-07	.119209E-06	.178814E-06
.500000E+00	.000000E+00	.119209E-06	.178814E-06
.700000E+00	.149012E-07	.119209E-06	.119209E-06
.900000E+00	.204891E-07	.894070E-07	.894070E-07
.950000E+00	.167638E-07	.596046E-07	.596046E-07

$$x_n(t) \cong \frac{-\sqrt{1-t^2}}{2\pi}(4-t) \qquad (3.44)$$

Which coincides with the exact solution. The error of approximate solution (3.44) of equation (3.40) at $n = 20$ is given by table 2.

CONCLUSION

Numerical results, show that the errors of approximate solutions of examples 1-6 with small value of n are very small. These show that the approximate method is very accurate.

The approximate method shows that the approximate solutions of the singular integral equations and the corresponding hypersingular integral equations are coincided.

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