SENRA Academic Publishers, Burnaby, British Columbia
Vol. 5, No. 2, pp. 1589-1595, June 2011
Online ISSN: 1920-3853; Print ISSN: 1715-9997

Canadian Journal of pure&applied sciences

SHORT COMMUNICATION

A POLYNOMIAL COLLOCATION METHOD FOR A CLASS OF NONLINEAR SINGULAR INTEGRAL EQUATIONS WITH A CARLEMAN SHIFT

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ABSTRACT

The paper is concerned with the applicability of the polynomial collocation method to a class of nonlinear singular integral equations with a Carleman shift preserving orientation on simple closed smooth Jordan curve in the generalized Holder space $H_{\varphi}(L)$. The method is illustrated by considering some simple examples.

2000 AMS: Subject classification 45G05, 47H10, 47H30, 45L10, 65R20.

Keywords: Nonlinear singular integral equations, carleman shift, polynomial collocation method, noetherian operator.

INTRODUCTION

Nonlinear singular integral equations are widely used and connected with applications in several field of engineering mechanics like structural analysis, fluid mechanics and aerodynamics. This leads to the necessity to derive solutions for the nonlinear singular integral equations arising in applications, by using some approximate and constructive methods, (Ladopoulous, 2000). The theory of nonlinear singular integral equations with Hilbert and Cauchy kernel and its related Riemann-Hilbert problems have been developed in works of Pogorozelski (1966), Guseinov and Mukhtarove (1980), Wolfersdorf (1985) and Ladopoulous (2000).

The successful development of the theory of singular integral equations (SIE) naturally stimulated the study of singular integral equations with shift (SIES). The Noether theory of singular integral operators with shift (SIOS) is developed for a closed and open contour (Kravchenko and Lebre, 1995; Kravchenko and Litvinchuk; 1994). Existence results and approximate solutions have been studied for nonlinear singular integral equations (NSIE) and nonlinear singular integral equations with shift (NSIES) by many authors among them we mention (Amer and Dardery (2004, 2005, 2009), Amer and Nagdy (2000), Amer (2001, 1996), Jinyuan (2000), Junghanns and Weber (1993), Ladopoulous and Zisis (1996), Ladopoulous (2000), Nguyen (1989) and Saleh and Amer (1992).

The classical and more recent results on the solvability of NSIE should be generalized to corresponding equations with shift (Wolfersdorf, 1992). The theory of SIES is an important part of integral equations because of its recent applications in many field of physics and engineering (Baturev *et al*., 1996; Kravchenko *et al*., 1995; Kravchenko and Litvinchuk, 1994).

We consider a simple closed smooth Jordan curve L in the complex plane with equation $t = t(s),\ 0 \le s \le l$ where s-arc coordinate accounts from some fixed point, l-length of the curve L. Denote by D^+ and D^- the interior and exterior domain of L respectively and let the origin be $0 \in D^+$. Denote by L_0 the unite circle with the center at the origin and let L_0^+ and L_0^- the interior and exterior domain of L_0 respectively. Consider the conformal mappings $A(r)$ from L_0^- onto D^- such that $A(\infty) = \infty$, $\lim\limits_{r\to\infty} A(r) r^{-1} > 0$ and $B(r)$ from L_0^- onto D^+ such that $B(\infty) = 0$.
Now, consider the following NSIES:

$$(\mathrm{P}(u))(t) = \Psi_1(t, u(t)) + \Psi_2(\alpha(t), u(\alpha(t))) - \\ -\frac{1}{\pi i}\int_L \left[\frac{\Psi_3(\tau, u(\tau))}{\tau - t} + \frac{\Psi_4(\tau, u(\tau))}{\tau - \alpha(t)}\right] \\ d\tau = f(t), \quad forall \quad t \in L \qquad (0.1)$$

Under the following conditions

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$$\psi_{1u}(t,u_o(t)) = \psi_{3u}(t,u_o(t)) = a(t),$$
$$\psi_{2u}(\alpha(t),u_o(\alpha(t))) = -\psi_{4u}(\alpha(t),u_o(\alpha(t))) = b(t). \quad (0.2)$$

for initial value u_0, in the generalized Holder space $H_\varphi(L)$, $u(t)$ is unknown function, $f(t)$ and $\Psi_r(t,u(t)),\ r=1,...,4$, are continuous functions on L and on the domain

$$D = \{(t,u): t \in L,\ u \in (-\infty,\infty)\},$$

respectively, and the homeomorphism $\alpha : L \to L$ is preserving orientation, satisfying the Carleman condition

$$\alpha(\alpha(t)) = \alpha_2(t) = t, \quad t \in L, \qquad (0.3)$$

and the derivative $\alpha'(t) \neq 0$ satisfies the usual Holder condition.

The equation (0.1) in case $f(t)=0$ without shift has been studied in Amer and Nagdy (2002) by modified Newton-Kantorovich method in the generalized Holder space $H_{\varphi,m}[a,b]$.

In this paper the polynomial collocation method has been applied to NSIES (0.1) under condition (0.2), with zero index, in the generalized Holder space $H_\varphi(L)$.

1. Some auxiliary results.

Definition 1.1. We denote by $H_{\varphi,1}(D)$ to be the space of all functions $\Psi_r(t,u(t)),\ r=1,...,4$, which have partial derivatives up to second order with respect to u and satisfy the following condition

$$\left|\psi_{ru^j}(t_1,u_1) - \psi_{ru^j}(t_2,u_2)\right| \le c_j^r \{\varphi(|t_1-t_2|) + |u_1-u_2|\}, \qquad (1.1)$$

where $(t_i,u_i) \in D$, $i=1,2,\ \varphi \in \Phi$ and c_j^r are constants; $j=0,1,2$.

Definition 1.2 (Guseinov and Mukhtarov, 1980; Mikhlin and Prossdorf, 1986). We denote by $c(L)$ the space of all continuous functions $u(t)$ defined on L with the norm:

$$\| u \|_{c(L)} = \max_{t\in L} |u(t)| . \qquad (1.2)$$

Definition 1.3 (Amer, 2001; Guseinov and Mukhtarov, 1980). We denote by $H_\varphi(L)$ the space of all functions $u(t) \in c(L)$ such that $\omega_u(\delta) = o(\varphi(\delta)),\ \varphi \in H\Phi$, with the norm:

$$\| u \|_\varphi = \| u \|_{c(L)} + \| u \| ; \qquad (1.3)$$

$$\| u \| = \sup_{\delta>0} \frac{\omega_u(\delta)}{\varphi(\delta)};$$

$$H\Phi = \left\{ \varphi \in \Phi : \int_0^\delta \frac{\varphi(\xi)}{\xi} d\xi + \delta \int_\delta^l \frac{\varphi(\xi)}{\xi^2} d\xi \le \tilde{c}\,\varphi(\delta) \right\},$$

$\tilde{c}$ is a positive constant.

Definition 1.4 (Amer, 2001; Kravchenko and Litvinchuk, 1994). Let $S : H_\varphi(L) \to H_\varphi(L)$ denotes to the operator of singular integration

$$(Su)(t) = \frac{1}{\pi i} \int_L \frac{u(\tau)}{\tau - t} d\tau , \qquad (1.4)$$

to which we associate the Cauchy projection operators

$$P_\pm = \frac{1}{2}(I \pm S), \quad S^2 = I , \qquad (1.5)$$

where I is the identity operator on $H_\varphi(L)$. The Carleman shift operator $W : H_\varphi(L) \to H_\varphi(L)$, is given by $(Wv)(t) = v(\alpha(t))$.

Lemma 1.1 (Amer, 2001). The singular operator S is a bounded operator on the space $H_\varphi(L)$ and satisfies the inequality

$$\|Su\|_\varphi \le \rho_0 \|u\|_\varphi, \qquad (1.6)$$

where ρ_0 is a constant defined as follows :

$$\rho_0 = c_1\left(\int_0^\delta \frac{\varphi(\xi)}{\xi} d\xi + 1 \right) + c_2 \tilde{c},$$

where c_1, c_2, $\tilde{c}$ are constants.

Lemma 1.2 (Amer, 2001). The shift operator W is a linear bounded continuously invertible operator on the space $H_\varphi(L)$ and satisfies the inequality

$$\| Wu \|_\varphi \le \gamma_0 \|u\|_\varphi , \qquad (1.7)$$

where $\gamma_0 = \max\{1,\alpha_0\}$ and α_0 is a constant given by

$$\alpha_0 = \sup_{\delta \succ 0} \frac{\omega_{\tilde{u}}(\delta)}{\omega_u(\delta)}, \quad \tilde{u}(t) = u(\alpha(t)).$$

Lemma 1.3 (Amer and Dardery, 2009) Let the functions $\psi_r(t,u),\ r=1,...,4$, belong to $H_{\varphi,1}(D)$. Then the operator $\mathrm{P}(u)$ is Frechet differentiable at every fixed point $u \in H_{\varphi}(L)$, moreover

$$\mathrm{P}'(u)h = \psi_{1u}(t,u(t))\,h(t) + \psi_{2u}(\alpha(t),u(\alpha(t)))\,h(\alpha(t)) - \\ -\frac{1}{\pi i}\int_L \left\{\frac{\psi_{3u}(\tau,u(\tau))}{\tau - t} + \frac{\psi_{4u}(\tau,u(\tau))}{\tau-\alpha(t)}\right\} h(\tau)\, d\tau, \qquad (1.8)$$

satisfies Lipschitz condition

$$\left\|\mathrm{P}'(u_1) - \mathrm{P}'(u_2)\right\|_{\varphi} \le \rho_1 \left\| u_1 - u_2 \right\|_{\varphi}, \qquad (1.9)$$

in the sphere $S_{\varphi}(u_0, r) = \left\{u \in H_{\varphi}(L) : \left\|u - u_0\right\|_{\varphi} \le r\right\}$,

where

$$\rho_1 = \left(c_1^1 + \gamma_0 c_1^2 + \rho_0 c_1^3 + \gamma_0 \rho_0 c_1^4\right).$$

Under condition (0.2), the equation (1.8) reduces to the following SIES, for the unknown function $h(t)$:

$$\Gamma_0 h = a(t)h(t) + b(t)h(\alpha(t)) - \frac{a(t)}{\pi i}\int_L \frac{h(\tau)}{\tau - t} d\tau + \\ + \frac{b(t)}{\pi i}\int_L \frac{h(\tau)}{\tau - \alpha(t)} d\tau + \frac{1}{\pi i}\int_L R(t,\tau) h(\tau)\, d\tau = f(t), \qquad (1.10)$$

for initial value u_o and the arbitrary function $f(t)$ belong to the space $H_{\varphi}(L)$,

where

$$\mathrm{R}(\mathrm{t},\tau) = \frac{\psi_{3\mathrm{u}}(\mathrm{t},\mathrm{u_o}(\mathrm{t})) - \psi_{3\mathrm{u}}(\tau,\mathrm{u_o}(\tau))}{\tau - \mathrm{t}} + \frac{\psi_{4\mathrm{u}}(\alpha(\mathrm{t}),\mathrm{u_o}(\alpha(\mathrm{t}))) - \psi_{4\mathrm{u}}(\tau,\mathrm{u_o}(\tau))}{\tau - \alpha(\mathrm{t})}.$$

Using Definition 1.4 the dominant equation of equation (1.10) reduces to the following singular integral operator with shift :

$$M = 2a(t)P_- + 2b(t)WP_+ . \qquad (1.11)$$

Theorem 1.1 (Amer and Dardery, 2009; Kravchenko and Litvinchuk, 1994). The singular integral functional operator M is Noetherian on $H_{\varphi}(L)$ if and only if

$$\inf\left|e(t)\right| > 0 \;\; and \;\; q(t) \ne 0, \; on\, L,$$

where

$$e(t) = 2b(t), q(t) = \frac{a(t)}{b(t)}; b(t) \ne 0 \;\; on\, L.$$

The index of a Noetherian operator M is given by

$$\chi = ind\, M = \frac{1}{2\pi}\left\{\arg q(t)\right\}_L . \qquad (1.12)$$

Theorem 1.2 (Amer 2001; Saleh and Amer, 1992). Let the conditions of Lemma 1.3 and Theorem 1.1 be satisfied and $u_0 \in H_{\varphi}(L)$ is the initial approximation for equation (0.1) under conditions (0.2), $\left\|\left(\mathrm{P}'(u_0)\right)^{-1}\right\|_{\varphi} \le \varepsilon_0$ and $\left\|\left(\mathrm{P}'(u_0)\right)^{-1}\mathrm{P}(u_0)\right\|_{\varphi} \le \varepsilon_1$. Then if $m = \varepsilon_0 \rho_1 \varepsilon_1 < 1/2$, then equation (0.1) under conditions (0.2) has a unique solution u^* in the sphere $S_{\varphi}(u_0; r_0)$ of the space $H_{\varphi}(L)$, $r_0 = \varepsilon_1\left(1 - \sqrt{1-2m}\right)m^{-1} \le r$, to which the successive approximations: $u_{n+1} = u_n - \left(\mathrm{P}'(u_0)\right)^{-1}\mathrm{P}(u_n)$ of modified Newton method converges and the rate of convergence is given by the inequality:

$$\left\|u_n - u^*\right\|_{\varphi} \le \frac{B^n}{1-B}\varepsilon_1; \;\; B = 1 - \sqrt{1-2m}$$

2. Collocation method.

Now, we seek an approximate solution of equation (0.1) under conditions (0.2) in $H_{\varphi}(L)$ as the form:

$$u_n(\eta, t) = \sum_{k=-n}^{n} \eta_k t^k , \qquad (2.1)$$

where the coefficients η_k are defined from the system of nonlinear algebraic equation with shift (SNAES)

$$\Psi_1(\mathrm{t_j}, \mathrm{u_n}(\eta, \mathrm{t_j})) + \Psi_2(\alpha(\mathrm{t_j}), \mathrm{u_n}(\eta, \alpha(\mathrm{t_j}))) \\ -\frac{1}{\pi \mathrm{i}}\int_{\mathrm{L}}\left[\frac{\Psi_3(\tau, \mathrm{u_n}(\eta,\tau))}{\tau - \mathrm{t_j}} + \frac{\Psi_4(\tau, \mathrm{u_n}(\eta,\tau))}{\tau - \alpha(\mathrm{t_j})}\right]d\tau = \mathrm{f}(\mathrm{t_j}). \qquad (2.2)$$

where $t_j = \exp(2\pi ij/(2n+1)),\; j = \overline{0,2n}$.

Consider (2n+1)- dimensional spaces $H_{\varphi}^{(1)}$ and $H_{\varphi}^{(2)}$ with the norms:

$$\left\|\eta\right\|_{\varphi}^{(1)} = \left\|u_n(\eta, .)\right\|_{\varphi},$$

$$\left\|u\right\|_{\varphi}^{(2)} = \max_j\left|u_j\right| + \sup_{j \ne k}\frac{\left|u_j - u_k\right|}{\varphi\left(\left|t_j - t_k\right|\right)},$$

respectively, where $\eta = (\eta_{-n}, ..., \eta_{-1}, \eta_0, ..., \eta_n) \in H_{\varphi}^{(1)}$ and $u = (u_0, ..., u_{2n}) \in H_{\varphi}^{(2)}$.

Introduce the operator $\mathrm{P}_n(\eta) : H_{\varphi}^{(1)} \to H_{\varphi}^{(2)}$ where

$$\mathrm{P_{j,n}}(\eta) = \Psi_1(\mathrm{t_j}, \mathrm{u_n}(\eta, \mathrm{t_j})) + \Psi_2(\alpha(\mathrm{t_j}), \mathrm{u_n}(\eta, \alpha(\mathrm{t_j}))) \\ -\frac{1}{\pi \mathrm{i}}\int_{\mathrm{L}}\left[\frac{\Psi_3(\tau, \mathrm{u_n}(\eta,\tau))}{\tau - \mathrm{t_j}} + \frac{\Psi_4(\tau, \mathrm{u_n}(\eta,\tau))}{\tau - \alpha(\mathrm{t_j})}\right]d\tau, \quad \mathrm{j} = \overline{0,2\mathrm{n}}$$

We can rewrite SNAES (2.2) in the operator form:

$$\mathrm{P}_n(\eta) = f; \quad f = f(t_j), \; j = \overline{0,2n}. \qquad (2.3)$$

Consider, the coordinates of the vector $\eta^{(0)}$ from $H_\varphi^{(1)}$ these are the Fourier coefficients of the function $u_0 \in H_\varphi(L)$ that is

$$\eta_j^{(0)} = \frac{1}{2\pi i}\int_{L_0} u_0(B(w))w^{-j-1}dw, \quad j=\overline{0,n} \quad \text{and}$$

$$\eta_j^{(0)} = \frac{1}{2\pi i}\int_{L_0} u_0(A(w))w^{-j-1}dw, \quad j=\overline{-n,-1}.$$

Analogous to Lemma 1.3 the following lemma is valid.

Lemma 2.1. Amer (1996) Let the conditions of Lemma 1.3 be satisfied. Then the operator P_n is Frechet differentiable at every fixed point $x=(\eta_{-n},...,\eta_n)\in H_\varphi^{(1)}$, Moreover

$$\mathrm{P}'_{j,n}(x)h = \psi_{1u}(t_j, u_n(x,t_j))\,u_n(h,t_j)$$
$$+\psi_{2u}(\alpha(t_j), u_n(x,\alpha(t_j)))\,u_n(h,\alpha(t_j)) -$$
$$-\frac{1}{\pi i}\int_L \left\{\frac{\psi_{3u_n}(\tau,u_n(x,\tau))}{\tau-t_j} + \frac{\psi_{4u_n}(\tau,u_n(x,\tau))}{\tau-\alpha(t_j)}\right\} u_n(h,\tau)\,d\tau, \quad j=\overline{0,2n}.$$

where $h=(h_{-n},...,h_n)\in H_\varphi^{(1)}$, the derivative $P_n'(x)=(P_{0,n}'(x),...,P_{2n,n}'(x))$ satisfies Lipschitz condition

$$\left\|\mathrm{P}_n{}'(x_1)-\mathrm{P}_n{}'(x_2)\right\|_{H_\varphi^{(2)}} \le \rho_1' \left\| x_1 - x_2 \right\|_{H_\varphi^{(1)}},$$

in the sphere $S(\eta^{(0)};r_1)$ of the space $H_\varphi^{(1)}$, where ρ_1' is a positive constant.

Now, we show that the system of linear algebraic equations with shift (SLAES):

$$\mathrm{P}_n'(\eta^{(0)})h = g, \tag{2.4}$$

under the conditions

$$\psi_{1u}(t_j,u_o(\eta^{(0)},t_j)) = \psi_{3u}(t_j,u_o(\eta^{(0)},t_j)) = a(t_j), \tag{2.5}$$
$$\psi_{2u}(\alpha(t_j),u_o(\eta^{(0)},\alpha(t_j))) =$$
$$-\psi_{4u}(\alpha(t_j),u_o(\eta^{(0)},\alpha(t_j))) = b(t_j).$$

has a unique solution $h\in H_\varphi^{(1)}$ for arbitrary $g=(g_0,...,g_{2n})\in H_\varphi^{(2)}$.

For this aim, we consider the SALES:

$$a(t_j)u_n(h,t_j)) + b(t_j)u_n(h,\alpha(t_j))\tau -$$
$$-\frac{a(t_j)}{\pi i}\int_L \frac{u_n(h,\tau)}{\tau-t_j}d\tau + \frac{b(t_j)}{\pi i}\int_L \frac{u_n(h,\tau)}{\tau-\alpha(t_j)}d\tau +$$
$$+\frac{1}{\pi i}\int_L R(t_j,\tau)u_n(h,\tau)\,d\tau = g(t_j), \quad j=\overline{0,2n} \tag{2.6}$$

corresponding to the SIES:

$$a(t)u(t)+b(t)u(\alpha(t)) - \frac{a(t)}{\pi i}\int_L \frac{u(\tau)}{\tau-t}d\tau + \frac{b(t)}{\pi i}\int_L \frac{u(\tau)}{\tau-\alpha(t)}d\tau +$$
$$+\frac{1}{\pi i}\int_L R(t,\tau)u(\tau)\,d\tau = g(t), \tag{2.7}$$

According to the collocation method, we seek an approximate solution of equation (1.10) as the form :

$$h_n(t) = \sum_{k=-n}^{n}\beta_k t^k, \quad t\in L, \tag{2.8}$$

where the coefficients β_k are defined from SLAES:

$$\sum_{k=-n}^{n} A_{jk}\beta_k = g(t_j), \quad j=\overline{0,2n} \tag{2.9}$$

where

$$A_{jk} = a(t_j)\left(t_j^k - \frac{1}{\pi i}\int_L \frac{\tau^k}{\tau-t_j}d\tau\right) + b(t_j)$$
$$\left((\alpha(t_j))^k + \frac{1}{\pi i}\int_L \frac{\tau^k}{\tau-\alpha(t_j)}d\tau\right) + \frac{1}{\pi i}\int_L R(t_j,\tau)h_n(\tau)d\tau$$

The SLAES (2.9) can be rewritten as following form:

$$2a(t_j)\sum_{k=-n}^{-1}\beta_k t_j^k + 2b(t_j)\sum_{k=0}^{n}\beta_k(\alpha(t_j))^k + \frac{1}{\pi i}\int_L R(t_j,\tau).$$
$$\sum_{k=-n}^{n}\beta_k\tau^k d\tau = g(t_j), \quad j=\overline{0,2n}. \tag{2.10}$$

Where

$$h_n^+(t) = \sum_{k=0}^{n}\beta_k t^k \quad , h_n^-(t) = -\sum_{k=-n}^{-1}\beta_k t^k,$$

Theorem 2.1. Let $a(t), b(t)$ and $g(t)$ belong to $H_\varphi(L)$, $b(t)\neq 0$ on L, the index $\chi=0$ and the operator P' has a linear inverse in $H_\varphi(L)$, then for all $n\ge \max(n_0,\chi)$,
$n_0 = \min\left\{n\in N : d_1\varphi\left(\frac{1}{n}\right)\ln n < 1\right\}$, the system (2.10) has the unique solution $\{\beta_k^*\}_{-n}^{n}$ and the approximate solution $, h_n^*(t) = \sum_{k=-n}^{n}\beta_k^* t^k$, of equation

(1.10) convergences to its exact solution h^*, moreover $\left\| h^*(t) - h_n^*(t) \right\|_\varphi \le d_2 \varphi\left(\frac{1}{n}\right) \ln n$,

where d_1 and d_2 are constants do not depend on n.

Proof.

From [Gakhov, 1966], we can write equation (1.10) in the following form:

$$h^+(\alpha(t)) - q(t)h^-(t) + \frac{1}{e(t)\pi i}\int_L R(t,\tau)h(\tau)d\tau = \frac{g(t)}{e(t)},$$

setting

$$q(t) = \frac{\psi^+(\alpha(t))}{\psi^-(t)}.$$

Then we have

$$\Gamma h = Bh + Gh = \tilde{g}\,. \tag{2.11}$$

Where

$$(Bh)(t) = \psi^-(t)h^+(\alpha(t)) - \psi^+(\alpha(t))h^-(t) \quad ,$$

$$(Gh)(t) = \frac{c(t)}{\pi i}\int_L R(t,\tau)h(\tau)d\tau\,,$$

$$\tilde{g}(t) = g(t)c(t),\; c(t) = \frac{\psi^-(t)}{e(t)},$$

$$\psi(z) = \exp(\theta(z)), \tag{2.12}$$

$$\theta(\mathrm{z}) = \frac{1}{\pi \mathrm{i}}\int_{\mathrm{L}} \frac{\rho(\gamma(\tau))}{\tau - \mathrm{z}} \mathrm{d}\tau;\ \mathrm{z} \in \mathrm{D}^+,$$

$$\theta(\mathrm{z}) = \frac{1}{\pi \mathrm{i}}\int_{\mathrm{L}} \frac{\rho(\tau)}{\tau - \mathrm{z}} \mathrm{d}\tau;\ \mathrm{z} \in \mathrm{D}^-\,,$$

where $\gamma(t)$ is the inverse $\alpha(t)$ and $\rho(t)$ is a solution of the Fredholm integral equation of second kind

$$\rho(t) + \frac{1}{\pi i}\int_L \left(\frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)} - \frac{1}{\tau - t} \right)\rho(\tau)d\tau = \ln q(t).$$

Moreover, B is linear and G is completely continuous from $H_\varphi(L)$ into itself.

Denote by X_n to be the (2n+1)- dimensional subspace of the space $H_\varphi(L)$, and let Q_n be the projection operator into the set of interpolation polynomial of degree n with respect to the collocation points $t_j,\ j = \overline{0,2n}$. Then the system (2.10) can be written in X_n as a linear operator

$$\Gamma_n h_n = B_n h_n + G_n h_n = \tilde{g}_n\,, \tag{2.13}$$

where

$$B_n h_n = Q_n B h_n\,,$$

$$G_n h_n = Q_n G h_n\,,\;\; \tilde{g}_n = Q_n \tilde{g}\,.$$

Now, we determine the difference $\Gamma h_n - \Gamma_n h_n \in X_n$, from (2.11), (2.13) we have

$$\begin{aligned}(\Gamma - \Gamma_n)h_n(t) = (I - Q_n)[(\psi^-(t) - \psi_n^-(t))h_n^+(\alpha(t)) - \\ - (\psi^+(\alpha(t)) - \psi_n^+(\alpha(t)))h_n^-(t)] + (G - G_n)h_n(t)\end{aligned} \tag{2.14}$$

where ψ_n is polynomial of the best uniform approximation of the function ψ with degree not exceeding n.

From [Amer, 1996, Gakhov, 1966] and inequality (1.7), we have

$$\left\| h_n^\pm \right\|_\varphi \le d_1 \left\| h_n \right\|_\varphi$$

$$\left\| \left[(\psi^-(t) - \psi_n^-(t))h_n^+(\alpha(t)) - (\psi^+(\alpha(t)) - \psi_n^+(\alpha(t)))h_n^-(t) \right] \right\|_\varphi \le \gamma_0 d_2 \varphi\left(\frac{1}{n}\right) \left\| h(t) \right\|_\varphi,$$

and

$$\left\| Q_n \right\|_\varphi \le d_3 \ln n\,.$$

Hence, we get

$$\left\| (I - Q_n)\left[(\psi^-(t) - \psi_n^-(t))h_n^+(\alpha(t)) - (\psi^+(\alpha(t)) - \psi_n^+(\alpha(t)))h_n^-(t) \right] \right\|_\varphi \le d_4 \varphi\left(\frac{1}{n}\right)(\ln n)\left\| h_n(t) \right\|_\varphi \tag{2.15}$$

where $d_4 = \gamma_0 d_2 d_3$.

Let $J_n(t)$ be the polynomial of best uniform approximation to the function

$$J(t) = \frac{c(t)}{\pi i}\int_L R(t,\tau)h_n(\tau)d\tau\,,$$

Then from Amer (1996), we have

$$\left\| J - J_n \right\|_\varphi \le d_5 \varphi\left(\frac{1}{n}\right) \left\| h_n \right\|_\varphi,$$

hence for arbitrary $h_n \in X_n$, we get

$$\left\| G h_n - G_n h_n \right\|_\varphi \le d_6 \varphi\left(\frac{1}{n}\right)(\ln n)\left\| h_n \right\|_\varphi, \tag{2.16}$$

where $d_6 \ln n = d_5 + d_3 d_5 \ln n$. From (2.14)- (2.16), we get

$$\left\| \Gamma h_n - \Gamma_n h_n \right\|_\varphi \le d_7 \varphi\left(\frac{1}{n}\right)(\ln n)\left\| h_n \right\|_\varphi, \tag{2.17}$$

where $d_7 = d_4 + d_6$. From Theorem 1.2 , the operator Γ_0 has a linear bounded inverse operator Γ_0^{-1} , since $\Gamma_0 h = c^{-1}\,\Gamma h$ then the operator Γ has a linear inverse, also from Amer (1996) and by virtue of (2.17) the operator Γ_n has a linear bounded inverse.

Now, for the right parts of (2.11) and (2.13), we have

$$\left\|\tilde{g}-\tilde{g}_n\right\|_\varphi \le d_8 \varphi\left(\frac{1}{n}\right)\ln n. \qquad (2.18)$$

From Amer (1996), and inequalities (2.17), (2.18) for the solution h^* of equation (1.10) and the approximate solution h_n^*, we obtain

$$\left\|h^*-h_n^*\right\|_\varphi \le d_9 \varphi\left(\frac{1}{n}\right)\ln n .$$

Thus the theorem is proved.

From Theorem 2.1 there exists the number n_0 such that for arbitrary $n \ge \max(n_0, \chi)$ the SLAES (2.6) has the unique solution h^* and the following inequality is valid:

$$\left\|u_n^*\left(h^*,.\right)-u^*(.)\right\|_\varphi \le d_{10} \varphi\left(\frac{1}{n}\right)\ln n \quad ,$$

where $u^* \in H_\varphi(L)$ is the unique solution of (2.7). Let

$$\Gamma_n(u_0)h = \left(\Gamma_{0,n}(u_0)h,\ldots,\Gamma_{2n,n}(u_0)h\right)$$

where

$$\Gamma_{n,j}(u_0)h = a(t_j)u_n(h,t_j)) + b(t_j)u_n(h,\alpha(t_j)) - \frac{a(t_j)}{\pi i}\int_L \frac{u_n(h,\tau)}{\tau - t_j}d\tau +$$
$$+\frac{b(t_j)}{\pi i}\int_L \frac{u_n(h,\tau)}{\tau-\alpha(t_j)}d\tau + \frac{1}{\pi i}\int_L R(t_j,\tau)u_n(h,\tau)\,d\tau, \quad j=\overline{0,2n}$$

From Amer (1996), we have

$$\left\|\Gamma_n(u_0) - \mathrm{P}_n^{'}\left(\eta^{(0)}\right)\right\|_{H_\varphi^{(1)}\to H_\varphi^{(2)}} \le d_{11}\varphi\left(\frac{1}{n}\right)\ln n . \quad (2.19)$$

Since for arbitrary $n \ge (n_0, \chi)$, there exists a bounded linear inverse operator, $\Gamma_n^{-1} : H_\varphi^{(2)} \to H_\varphi^{(1)}$ then from (2.19), Banach theorem follows that there exists $n_1 \ge (n_0, \chi)$ such that for arbitrary $n \ge n_1$, the linear operator $\mathrm{P}'_{j,n}$ has bounded inverse, that is the SLAES (2.4) under condition (2.5) has the unique solution $h^* \in H_\varphi^{(1)}$ for arbitrary right side $g = g(t_j) \in H_\varphi^{(2)}$, $j=\overline{0,2n}$. Thus the following theorem is proved.

Theorem 2.2 Let the coordinate of the vector $\eta^{(0)} = \left(\eta_{-n}^{(0)},\ldots,\eta_{-1}^{(0)},\eta_0^{(0)},\ldots,\eta_n^{(0)}\right)$ be the Fourier coefficients the function $u_0 \in H_\varphi(L)$ and the conditions of Theorem 1.2 are satisfied and for $n \ge n_1$, $\left\|\left(\mathrm{P}'_n\left(\eta^{(0)}\right)\right)^{-1}\right\|_\varphi \le \varepsilon'_0$ and $\left\|\left(\mathrm{P}'_n\left(\eta^{(0)}\right)\right)^{-1}\mathrm{P}_n\left(\eta^{(0)}\right)\right\|_\varphi \le \varepsilon'_1$. Then if $m' = \varepsilon'_0\rho'_1\varepsilon'_1 < 1/2$, then SNAES (2.3) has the unique solution $\eta^* = \left(\eta_{-n}^*,\ldots,\eta_{-1}^*,\eta_0^*,\ldots,\eta_n^*\right)$ in the sphere $S_\varphi\left(\eta^{(0)}; r'_0\right)$ of the space $H_\varphi(L)$, $r'_0 = \varepsilon'_1\left(1-\sqrt{1-2m'}\right)(m')^{-1} \le r'$, to which the following iteration process converges $\eta^{(m+1)} = \eta^{(m)} - \left(\mathrm{P}'_n\left(\eta^{(0)}\right)\right)^{-1}\mathrm{P}_n\left(\eta^{(m)}\right)$ and the rate of convergence is given by the inequality: $\left\|\eta^{(m)}-\eta^*\right\|_\varphi \le \frac{B_1^n}{1-B_1}\varepsilon'_1$; $B_1 = 1-\sqrt{1-2m'}$.

3. Illustrative examples

We illustrate the above method by some problems.

Example 1.

Consider the following integral equation

$$t^2h(t) - -\frac{1}{\pi i}\int_L \frac{h(\tau)}{\tau - t}d\tau = t^3 + t \qquad (3.1)$$

Where the contour L is a unit circle in the complex plane.

It is easy to find that the index of equation (3.1) equal to zero and the exact solution takes the form $h(t) = t.$

According to the collocation method the approximate solution of equation (3.1) takes the form (2.8), where the coefficients β_k are defined from SLAE

$$\left(a(t_j)+b(t_j)\right)\sum_{k=0}^{n}\beta_k t_j^k + \left(a(t_j)-b(t_j)\right)\sum_{k=-n}^{-1}\beta_k t_j^k = g(t_j), \ j=\overline{0,2n}, \qquad (3.2)$$

where

$$t_j = \exp(2\pi i j/(2n+1)), \quad a(t_j) = t_j^2, \quad b(t_j) = -1, \quad g(t_j) = t_j^3 + t_j \qquad (3.3)$$

From relation (3.3) we get

$$\left(t_j^2-1\right)\sum_{k=0}^{n}\beta_k t_j^k + \left(t_j^2+1\right)\sum_{k=-n}^{-1}\beta_k t_j^k = t_j^3 + t_j, \quad j=\overline{0,2n} \qquad (3.4)$$

By solving SLAE (3.4) we found the approximate solution takes the form $h_n(t) = t$ for $n \ge 2$.

Example 2.

Consider the following integral equation

$$t\,h(t)-\frac{(t-2)}{\pi i}\int_L \frac{h(\tau)}{\tau-t}d\tau = 2\left(t^2-1\right) \qquad (3.5)$$

where the contour L is the circle $|z| = 1/2$ in the complex plane.

It is easy to find that the index of equation (3.5) equal to zero and the exact solution takes the form $h(t) = t^2 - 1.$ According to the collocation method the approximate solution of equation (3.1) takes the form (2.8), where the coefficients β_k are defined from SLAE (3.2);

$$a(t_j) = t_j,\;\; b(t_j) = 2 - t_j,\;\; g(t_j) = 2(t_j^2 - 1) \quad (3.6)$$

From relation (3.6) we get

$$\sum_{k=0}^{n} \beta_k t_j^k + (t_j - 1)\sum_{k=-n}^{-1} \beta_k t_j^k = t_j^2 - 1, \quad j = \overline{0,2n} \qquad (3.7)$$

By solving SLAE (3.7) we found the approximate solution coincides with the exact solution for $n \geq 2$.

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Received: Jan 17, 2011; Accepted: March 21, 2011