Canadian Journal of Pure and Applied Sciences
Vol. 18, No. 1, 2, 3, pp. 5859-5863, Oct 2024
Online ISSN: 1920-3853; Print ISSN: 1715-9997
Available online at www.cjpas.net

CHARACTERIZATION OF CYCLIC GROUPS VIA THE SIZE OF GENERATORS

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ABSTRACT

Cyclic groups have been characterised in several ways by so many authors over time, but none of these characterizations have been directly in respect of the main ingredient for which the cyclic nature of a group is hinged on; the generator of the group. In this paper therefore, we feel the need to cover this gap, hence, we characterize a cyclic group by the number of its generators. We achieve this by introducing the notion of 'level of a cyclic group', denoted $l(G)$, given a cyclic group G. With this new characterization, we obtain some important results on cyclic groups up to isomorphism. In addition, we introduce the notion of the 'cyclicity degree of a cyclic group' using $l(G)$ and then present some non trivial examples to support our main results.

Keywords: Euler Phi function, generators, cyclic groups, level of a cyclic group, cyclicity degree.

INTRODUCTION

A group is cyclic so long as it has at least one element called its generator that generates every other element in it. Several works have been carried out on the characterization of cyclic groups dating back to the 80's and 90's up to the present decade. This is not unexpected considering its significance. In fact, Kovacs (2000) characterized cyclic groups by sums and differences. Lanski (2001) characterized infinite cyclic groups, while a characterisation of finite cyclic groups was given by Walls (2004).

Oman and Slattum (2016) characterized cyclic groups via subgroup indices. In their work they showed that a certain property (distinct subgroups of a finite cyclic group has distinct cardinalities, which they tagged (D)) enjoyed by the cyclic groups completely distinguishes them within the class of all groups. Hence, they showed that a group has the property (D) if distinct subgroups of G have distinct indices in G. Further, while characterizing cyclic groups via indices of maximal subgroups, Allan (2022) showed that cyclic groups are the only finitely generated groups with the property that distinct maximal subgroups have distinct indices.

In all of the characterizations in published literature, none has directly captured the essence of the cyclic nature of a group, in other word there has not being any characterization of cyclic groups in terms of its generator. This is what we present in this paper, as we characterize a cyclic group by the size (magnitude or number) of its generators. This leads us to the introduction of the notion of level of a cyclic group, which we denote $l(G)$, given a cyclic group G. Hence a cyclic group is said to be of level n if it has n number of generators.

In 2009, Tarnauceanu introduced and studied the subgroup commutativity degree of finite groups. He added to the study in Tarnauceanu (2011). Later in Tarnauceanu (2016), the subgroup commutativity degree of finite p- groups was studied while the normality degree of finite cyclic groups was introduced and studied in Tarnauceanu (2017). Eniola (2019) introduced the commutativity degree of profinite groups, where a probability measure which counts the pairs of closed commuting subgroups in infinite groups was found. The measure turned out to be an extension of what was known in the finite case as a subgroup commutativity degree. The relative 2- Engel degree of a subgroup of a finite group was studied by Haniel and Abbas (2022) which was aimed at defining the probability of two randomly chosen elements satisfying the 2- Engel condition $[y,_2 x] = 1$ given two elements x and y.

Inspired by Tarnauceanu (2009), Tarnauceanu and Lazlo (2015) introduced and studied the notion of cyclicity degree of a finite group G, a quantity that measures the

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probability of a random subgroup of G to be cyclic. The authors noted that for an arbitrary finite group, G, computing the number of its subgroups as well as the number of cyclic subgroups is a difficult task as these numbers are in general unknown only for few particular classes of finite groups.

In view of this constraint, it is natural to ask how cyclic can a cyclic group be, and whether it is possible to calculate the cyclicity degree of a cyclic group using its own generators. Motivated and inspired by the results in Tarnauceanu and Lazlo (2015) and other results in this direction, in this paper, we give answers to the first question. We achieve this by the introduction of the notion 'Level of a Cyclic Group', a quantity that indicates the size of generators a given cyclic group has.

The rest of the paper is organized as follows: in section two, some basic definitions, properties and known results that would be required to establish our main results in sections three and four are presented. In section three, the definition of the level of a cyclic group and some useful results based on this characterization are given. Finally, in section four, the cyclicity degree of a cyclic group is defined in terms of this new characterization and results obtained in section three are then used to obtain the cyclicity degree of some cyclic groups.

Our result as compared to those obtained in Tarnauceanu and Lazlo (2015) is quite obvious. While Tarnauceanu and Lazlo (2015) obtained the cyclicity degree of a finite group as a result that checks if a given subgroup of a finite group is cyclic, we find the cyclicity degree of an 'already' cyclic group. Hence, by this, we answer the question as to how cyclic a cyclic group could be. Finally, we presented a possible area of application of this method of characterization of cyclic groups.

2 Preliminaries

The following definitions will be needed to establish our results:

Definition 2.1 (Fraleigh, 1968): A group G is cyclic if in G there is an element x, such that $G = \{x^n : n \in \mathbb{Z}\}$. Such element, x is called the generator of G. When x generates G, we write $G = \langle x \rangle$.

Definition 2.2 (Fraleigh, 1968): The number of elements in a group G is known as the order of G and is denoted $|G|$.

Definition 2.3 (Fraleigh, 1968): Let G be a group. The order of an element $x \in G$ is the least positive integer n such that $x^n = e$, where e is the identity element of G and the binary operation on G is defined multiplicatively. When the binary operation on G is additive, we define the order of $x \in G$ as the least positive integer m such that $mx = e$.

Definition 2.4 (Fraleigh, 1968): If a cyclic group G is of finite order and $x \in G$, then the order of x, denotedby $|\langle x \rangle|$, is the order of the cyclic group. Otherwise, we say that x is of infinite order.

Definition 2.5 (Fraleigh, 1968): Two positive integers m, n are said to be relatively prime if their greatest common divisor (GCD) is 1, i.e. m, n are relatively prime if $(m, n) = 1$, where (m, n) stands for the GCD of m and n.

Definition 2.6 (Benjamin, and Gerhard, 2007): For any integer $n > 0$,

$\Phi(n) = number\ of\ integers\ less\ than\ or\ equal\ to\ n\ and\ relatively\ prime\ to\ n.$

$\Phi(n)$ is called the Euler Phi function.

The following results are very useful, we shall make reference to them in time:

Theorem 2.7 (Fraleigh, Corollary. 6.16): If x is a generator of a finite cyclic group G of order n, then the other generators of G are the elements of the form x^r, where r is relatively prime to n.

Theorem 2.8 (Fraleigh, Theorem 6.10): Let G be a cyclic group with generator x. If the order of G is infinite, then G is isomorphic to $(\mathbb{Z}, +)$, the cyclic group of the integers with respect to the binary operation of addition. If G is of finite order, then it is isomorphic to $(\mathbb{Z}_n, +)$, the group of integer modulo n with respect to additionmodulo n.

Remark 2.9: The above result (Theorem 2.8) hereby classifies all cyclic groups into two types up to isomorphism, viz. $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ for infinite and finite cyclic groups, respectively. In this paper therefore, we shall center all our work on these two forms of cyclic groups.

Now, we present the definition of the cyclicity degree of a finite group due to Tarnauceanu and Laszlo (2015).

Definition 2.10: Let G be a finite group. Then the cyclicity degree of G, denoted by $cdeg(G)$, is given by

$$cdeg(G) = \frac{|C(G)|}{|L(G)|},$$

where $|C(G)|$ is the order of the poset of the cyclic subgroup of G, and $|L(G)|$ denotes the order of the subgroup lattice of G.

3 Level of a cyclic group

In this section, we characterize a cyclic group via the size of its generators and then obtain some useful results. We begin by introducing the notion of Level of a cycle group.

Definition 3.1: Let G be a cyclic group. We define the Level of G as the number of generators contained in G. It is denoted $l(G)$.Now, suppose G is a cyclic group of order n. If G has only one generator, then $l(G) = 1$, and G is called a Level 1 cyclic group. If G has only two generators, then $l(G) = 2$, and G is called a Level 2 cyclic group, and so on.

Theorem 3.2: Let G be a cyclic group. Then the level of G is greater than or equal to one, i.e. $l(G) \geq 1$.

Proof: Since a cyclic group must have at least one generator, the result follows.

Remark 3.2.1: This result covers for both the finite and infinite cyclic groups.

The following theorem only applies for the case in which a group is finite.

Theorem 3.3: Let G be a cyclic group of order n. Then $1 \leq l(G) < n$.

Proof: By theorem 2.8 in section two, every finite cyclic group is isomorphic to $(\mathbb{Z}_n, +)$.Since $0 \in \mathbb{Z}_n$ is not a generator, $(\mathbb{Z}_n, +)$ has less than n generators. This implies that any finite cyclic group$G$ of order n has less than n generators. Combining this with the result in theorem 3.2 above, we have that $1 \leq l(G) < n$. Hence, the proof is completed.

Next, we present the following propositions:

Proposition 3.4: Let $n \in \mathbb{Z}^+, n > 1$. Then the cyclic group $(\mathbb{Z}_n, +)$ is a level $\Phi(n)$ cyclic group if n is composite.

Proof: It is obvious that for $n \in \mathbb{Z}^+, n > 1$ and n composite, the generators of the cyclic group $(\mathbb{Z}_n, +)$ are the integers in $\mathbb{Z}_n$ that are relatively prime to n. Hence, there are $\Phi(n)$ generators of the cyclic group. Therefore, $(\mathbb{Z}_n, +)$ is a Level $\Phi(n)$ cyclic group.

Proposition 3.5: The cyclic group $(\mathbb{Z}, +)$ is a Level 1 cyclic group.

Proof: It is obvious.

Proposition 3.6: Let $n \in \mathbb{Z}^+, n > 1$. Then the cyclic group $(\mathbb{Z}_n, +)$ is a Level $(n-1)$ cyclic group if n is prime.

Proof: Since every member of $\mathbb{Z}_n$(n prime), except 0 is relatively prime to n, the result follows from proposition 3.4.

Now, we give the following example to support the above results.

Example 3.7: The cyclic group $U_4 = \{1, i, -1, -i\}$of the fourth root of unity with respect to complex number multiplication is a Level 2 cyclic group.

Proof: Since U_4 has i and $-i$ as generators, the result follows.

This example is a clear application of proposition 3.4.

As a final piece in this section, we take a necessary deviation to the group of nonzero integers modulo n with respect to multiplication modulo n. Of course, it is only a group when n is prime and it is cyclic.

Theorem 3.8: Let $n = p$ (a prime) and $G = (\mathbb{Z}_p \backslash \{0\}, .)$ be the cyclic group of nonzero integers modulo $p$with respect to multiplication modulo p. Then, $l(G) = |\mathbb{Z}_p \backslash \{0\}| - \eta(E(\mathbb{Z}_p))$, where $\eta(E(\mathbb{Z}_p))$ is the number of even members in $\mathbb{Z}_p$. Thus, G is a Level $(|\mathbb{Z}_p \backslash \{0\}| - \eta(E(\mathbb{Z}_p)))$ cyclic group.

Proof: The proof follows from the fact that all elements of the nonzero cyclic group $\mathbb{Z}_p$with respect to multiplication modulo p are relatively prime to p but not all are generators of $\mathbb{Z}_p$. In fact all the even integers in $\mathbb{Z}_p$ are not generators. Hence, the number$l(G) = |\mathbb{Z}_p \backslash \{0\}| - \eta(E(\mathbb{Z}_p))$, consequently, $G = (\mathbb{Z}_p, .)$ is a Level $(|\mathbb{Z}_p \backslash \{0\}| - \eta(E(\mathbb{Z}_p)))$ cyclic group.

4. Cyclicity degree of a cyclic group

In this section, we now describe the cyclicity degree of a cyclic group using the new characterization presented in section three, hence, presenting a result analogous to the results obtained in Tarnauceanu and Lazlo (2015) using a different approach. We then proceed to find the cyclicity degree of each of the cyclic groups whose Levels were obtained in section three. First, we define the cyclicity degree of a cyclic group.

Definition 4.1: Let G be a finite cyclic group. We define the cyclicity degree of G, $cdeg(G)$ of G as follows:

$$cdeg(G) = \frac{Level\ of\ G}{Order\ of\ G} = \frac{l(G)}{|G|}.$$

$cdeg(G)$ takes values in the open interval $(0, 1)$. This quantity measures the strength of the Level of any finite cyclic group on the 0 to 1 scale. It could as well be obtained as a percentage.

Example 4.2: Let $n \in \mathbb{Z}^+, n > 1$ and n composite. Then the cyclicity degree of the cyclic group $G = (\mathbb{Z}_n, +)$ is given by

$$cdeg(G) = \frac{\Phi(n)}{n},$$

since G is a level $\Phi(n)$ cyclic group if n is composite, where $\Phi(n)$ is the Euler Phi function.

As a direct example, the cyclicity degree of U_4 in proposition 3.7 is$\frac{1}{2}$.

Example 4.3: Let $n \in \mathbb{Z}^+, n > 1$ and n prime. Then the cyclicity degree of the cyclic group $G = (\mathbb{Z}_n, +)$ is given by

$$cdeg(G) = \frac{n-1}{n} = 1 - \frac{1}{n},$$

since G is a Level $(n-1)$ cyclic group if n is prime.
As a direct example, the cyclicity degree of $(\mathbb{Z}_5, +)$ is $\frac{4}{5}$.

Example 4.4: Let $n = p$ (a prime integer) and $G = (\mathbb{Z}_p, .)$ be the cyclic group of nonzero integers modulo p with respect to multiplication modulo p. Then the cyclicity degree of G is given by

$$cdeg(G) = \frac{|\mathbb{Z}_p \backslash \{0\}| - \eta(E(\mathbb{Z}_p))}{p-1},$$

since G is a Level $|\mathbb{Z}_p \backslash \{0\}| - \eta\left(E\left(\mathbb{Z}_p\right)\right)|$cyclic group.

As a direct example, the cyclicity degree of $(\mathbb{Z}_5, .)$ is $\frac{3}{4}$.
The difference between this result and the earlier result obtained when $G = (\mathbb{Z}_5, +)$ is very clear.

Application

Consider a game designed in such a way that a player is expected to hit a score that triggers other scores in the game, which in turn alerts the jackpot. In other words, to hit the jackpot, one has to hit a number that could trigger others. This number is the generator of the others. Hence, for a game to be highly competitive, it must be designed with the idea of a low Level cyclic group. This in turn implies a low cyclicity degree.

CONCLUSION

In this work we have characterised a cyclic group by the size of its generators by the help of the notion of Level of a cyclic group. Consequently, various results on cyclic groups up to isomorphism obtained. Using this new approach, the cyclicity degree of some cyclic groups were then calculated with the results obtained showing a direct relationship between the Level of a cyclic group and its degree of cyclicity.

ACKNOWLEDGEMENT

I sincerely want to appreciate the anonymous reviewer's patience and expertise in reviewing this work. We also want say a big thank you to my friend and brother, Mr. Imo I.K Agwu for his encouragements.

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Received: June 18, 2024; Revised: August 9, 2024;
Accepted: September 30, 2024